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1992 J. Phys. A: Math. Gen. 25 L1295

(http://iopscience.iop.org/0305-4470/25/23/005)

J. Phys. A: Math. Gen. 25 (1992) L1295–L1298. Printed in the UK

LETTER TO THE EDITOR

Expectation values of squeezing Hamiltonians in the SU(1, 1) coherent states

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Received 17 August 1992

Abstract. The classical discrete dynamics generated by the squeezing Hamiltonian can be extracted from the expression for the path integral over the SU(1, 1) coherent states in the continuous limit. In this letter the general formulae for the expectation values of the Hamiltonians of this type in these states are presented.

The purpose of this letter is to find general expressions for the expectation values of the products of positive integer powers of the SU(1, 1) generators in the SU(1, 1) (generalized) coherent states. Formulae of this type may be useful in the investigation of the classical dynamics generated by various types of squeezing Hamiltonians (cf e.g. Gerry and Vrscay 1989, Gerry and Kiefer 1990, Gerry *et al* 1991, and Bechler and Lisowski 1991).

The interaction of the coherent light with the nonlinear optical medium can be described by model Hamiltonians of various forms. In the simplest case a Hamiltonian of this type could be written as (Yuen 1976)

$$H = \tfrac{1}{2}\hbar\omega(a^\dagger a + aa^\dagger) + \tfrac{1}{2}\hbar\chi a^{\dagger 2} + \tfrac{1}{2}\hbar\chi^* a^2 \tag{1}$$

where $[a, a^\dagger] = 1$.

We can introduce three operators

$$K_0 = \tfrac{1}{4}(a^\dagger a + aa^\dagger) \qquad K_+ = \tfrac{1}{2}a^{\dagger 2} \qquad K_- = \tfrac{1}{2}a^2 \tag{2}$$

satisfying the commutation relations

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_-, K_+] = 2K_0 \tag{3}$$

which generate the Lie algebra of the SU(1, 1) group (Gerry and Silverman 1982). The Hamiltonian (1) is then linear in these operators

$$H = 2\hbar\omega K_0 + \hbar\chi K_+ + \hbar\chi^* K_-.$$

An example of a nonlinear Hamiltonian describing a single mode of the electromagnetic field was given by Tombesi and Mecozzi (1988)

$$H = 4\gamma^{(3)}(K_+K_- + K_-K_+) - 4\gamma^{(4)}E(K_-^2 + K_+^2)$$

and in a simpler form by Gerry and Kiefer (1990)

$$H = 2\omega K_0 + \tfrac{1}{2}\lambda(K_-^2 + K_+^2). \tag{4}$$

0305-4470/92/231295+04$07.50

There is a large class of squeezing Hamiltonians that can be expressed in terms of the SU(1, 1) generators. The definition of these generators might then be different from (2), but the commutation relations (3) have to be conserved. In this case one can use (3) to transform the Hamiltonian into the linear combination of products of K_0, K_+ and K_- in the ordered form

$$h_{p,q,r} = K_-^p K_0^q K_+^r \tag{5}$$

where p, q and r are non-negative integers. Next we will present the formulae for the expectation values of $h_{p,q,r}$ in the SU(1, 1) coherent states. Because we consider only squeezing Hamiltonians, that are linear in $h_{p,q,r}$, this allows us to build the expectation value of any Hamiltonian of this kind.

We can use the definition of the SU(1, 1) coherent states given by Perelomov (1972). We choose the irreducible representation $\mathscr{D}^+(k)$, the positive discrete series (Barut and Fronsdal 1965), where the operator K_0 is diagonal, i.e.

$$K_0|k, n\rangle = (k+n)|k, n\rangle \tag{6}$$

where $|k, n\rangle$ are the basis vectors of this representation ($n = 0, 1, 2, \ldots$ and $k > 0$). The SU(1, 1) coherent states are constructed by the unitary operator $D(\alpha)$ acting on the vector $|k, 0\rangle$ (Gerry and Silverman 1982), where

$$D(\alpha) = \exp(\alpha K_+ - \alpha^* K_-) \qquad \alpha = -(\vartheta/2)\,\mathrm{e}^{-\mathrm{i}\phi}.$$

Using the Lie algebra (3) we get the following form of the SU(1, 1) coherent state

$$|\xi, k\rangle = (1-|\xi|^2)^k \exp(\xi K_+)|k, 0\rangle \tag{7}$$

where

$$\xi = -\tanh(\vartheta/2)\,\mathrm{e}^{-\mathrm{i}\phi} \tag{8}$$

is the SU(1, 1) group parameter.

Let us recall the formula expressing the vector $|k, n\rangle$ by $|k, 0\rangle$ (Gerry and Silverman 1982)

$$|k, n\rangle = \left(\frac{\Gamma(2k)}{n!\Gamma(n+2k)}\right)^{1/2} (K_+)^n|k, 0\rangle. \tag{9}$$

We are primarily interested in the expectation values of (5) in the SU(1, 1) coherent states

$$M_{p,q,r} = \langle \xi, k|K_-^p K_0^q K_+^r|\xi, k\rangle. \tag{10}$$

Using (7) and (9) we can express the action of operators K_+^r and K_-^p on the coherent state $|\xi, k\rangle$ and its Hermitian conjugate as a series of basis vectors of the $\mathscr{D}^+(k)$ representation. K_0^q acts on these states according to (6), producing additional factors. Finally using the orthogonality properties of the $|k, n\rangle$ states we get the following formula

$$M_{p,q,r} = (1-x)^{2k}\xi^{p-r} \sum_{n=0}^{\infty} \frac{\Gamma(n+p+1)\Gamma(n+p+2k)}{n!\Gamma(n+p+1-r)\Gamma(2k)}(n+p+k)^q x^n \tag{11}$$

where $x = |\xi|^2$.

If $p < r$, the gamma function in the denominator has simple poles for the first few values of n. As a result, the sum effectively begins in this case at $n = r - p > 0$.

The sum in (11) depends on x and can be expressed as an action of a differential operator on the expression

$$(1-x)^{-\alpha}=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}x^{n} \tag{12}$$

for some real α. The differential operator should be constructed in such a way that it would produce all additional factors in (11) not present in (12). This approach leads us to two equivalent closed formulae

$$M_{p,q,r}=\frac{\Gamma(p+2k)}{\Gamma(2k)}(1-x)^{2k}(\xi^*)^{r-p}\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left[\frac{1}{x^k}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^q\frac{x^{p+k}}{(1-x)^{p+2k}}\right] \tag{13}$$

$$M_{p,q,r}=\frac{\Gamma(p+2k)}{\Gamma(2k)}\frac{(1-x)^{2k}}{\xi^{r+k}\xi^{*p+k}}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^q\left[x^{r+k}\frac{\mathrm{d}^r}{\mathrm{d}x^r}\frac{x^p}{(1-x)^{p+2k}}\right]. \tag{14}$$

From (11) we can also derive recurrence relations which are sometimes more convenient than the closed, but still complex, expressions given above

$$M_{p,q,r}=\xi^*M_{p+1,q-1,r}+(k+p)M_{p,q-1,r} \tag{15a}$$

$$M_{p,q,r}=\frac{1}{\xi}(\xi^*M_{p+1,q,r-1}+(p+1-r)M_{p,q,r-1}) \tag{15b}$$

$$M_{p,0,0}=\frac{\Gamma(p+2k)}{\Gamma(2k)}\left(\frac{\xi}{1-|\xi|^2}\right)^p. \tag{16}$$

One can use the relations (15a) and (15b) to reduce $M_{p,q,r}$ to a combination of $M_{p,0,0}$ with different p, given by (16).

The Hamiltonians constructed of K_-, K_+ and K_0 usually contain pairs of terms, which are mutually Hermitian conjugates. This leads to the expectation values $M_{p,q,r}$ and $M_{r,q,p}$ (note the reversed order of indices). From (11) we can find the following relation

$$M_{r,q,p}=M^*_{p,q,r}=M_{p,q,r}\left(\frac{\xi^*}{\xi}\right)^{p-r}. \tag{17}$$

As an example we can investigate a simple generalization of the model Hamiltonian (4)

$$H=2\omega K_0+\tfrac{1}{2}\lambda(K_-^n+K_+^n). \tag{18}$$

Using (15a), (16) and (17) we find the expectation value of the Hamiltonian (18) in the state $|\xi,k\rangle$

$$\langle\xi,k|H|\xi,k\rangle=2\omega k\cosh\vartheta+\left(-\frac{1}{2}\right)^n\lambda\frac{\Gamma(n+2k)}{\Gamma(2k)}\sinh^n\vartheta\cos(n\phi) \tag{19}$$

where we used the expression (8) for the ξ parameter. Following the method described by Gerry and Kiefer (1990) we find the equation of motion for the variables ϑ and ϕ

$$\dot{\vartheta}=n\lambda\left(-\frac{1}{2}\right)^n\frac{\Gamma(n+2k)}{k\Gamma(2k)}\sinh^{n-1}\vartheta\sin(n\phi) \tag{20a}$$

$$\dot{\phi}=2\omega+n\lambda\left(-\frac{1}{2}\right)^n\frac{\Gamma(n+2k)}{k\Gamma(2k)}\sinh^{n-2}\vartheta\cosh\vartheta\cos(n\phi). \tag{20b}$$

For brevity we can call the common factor

$$C_n = \frac{n\Gamma(n+2k)}{2^n k\Gamma(2k)}.$$

The stationary points of this system of equations are defined by the conditions: $\dot{\vartheta} = 0$ and $\dot{\phi} = 0$.

The first condition is satisfied if $\sin(n\phi) = 0$ or $\sinh\vartheta = 0$ (for $n > 1$). We can see from (20*b*) that the second possibility cannot be satisfied for $n > 2$, since usually $\omega \neq 0$. As we will see, among all possible values of n the case of $n = 2$ is particular and has special properties.

The condition $\sin(n\phi) = 0$ gives us a family of solutions $\phi_s = m\pi/n$, $m \in \mathbb{Z}$. In this case the equation $\dot{\phi} = 0$ takes the form

$$2\omega = (-1)^{n+m+1}\lambda C_n \sinh^{n-2}\vartheta \cosh\vartheta. \tag{21}$$

Among all integer values of m, for n even the solution can exist only if m is odd (for n odd there is no such limitation). For $n > 2$ the solution can exist for any positive values of λ and ω, but for $n = 2$ there is a critical value of λ ($\lambda_c = 2\omega/C_2$), above which the solution of (21) does not exist ($\cosh\vartheta \geqslant 1$). In the latter case, when $\lambda > \lambda_c$ the only stationary point can be found for $\sinh\vartheta = 0$ and $\cos(2\phi) = -\lambda_c/\lambda$. Gerry and Kiefer (1990) showed, that for $n = 2$, the energy surface changes when one passes from $\lambda < \lambda_c$ to $\lambda > \lambda_c$. This is associated with the first-order phase transition in the ground-state energy. From our brief discussion we can see that in the case of the general Hamiltonian (18), the phase transition in the ground-state energy occurs only for $n = 2$ (Gerry and Kiefer 1990). For other n we observe only rotational orbits on the energy surface.

The systems where the Hamiltonian can be written as a linear combination of the SU(1, 1) group generators, appear in the classical picture as an oscillator in the space of the ξ parameter (Lobachevsky plane) (Gerry and Silverman 1982, Bechler and Lisowski 1991). The formulae for the expectation values of the Hamiltonian in the SU(1, 1) coherent states presented in this letter may help to investigate more complex, nonlinear systems, where the added anharmonicity is expected to produce chaotic behaviour in the classical picture of discrete dynamics.

I wish to thank Professor A Bechler for stimulating conversations and for bringing to my attention the paper by Gerry and Kiefer (1990), from which developed the idea for the example presented here.

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